# Simplified Laplace Transform Inversion for Unsteady Surface Element Method for Transient Conduction

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**The unsteady surface element method is an analytical procedure for solving certain types of transient heat conduction problems. The method is intended for thermally contacting bodies of similar or dissimilar geometries such as occur in the contact conductance, fin, and temperature measurement problems. The method utilizes Duhamel's integral. The two simplified procedures in this paper utilize an approximate inverse Laplace transform relation given by Schapery to obtain functional forms of the solutions. In certain cases the approximate inverse Laplace relations simplify the solution greatly and produce acceptable accuracy. Tauberian, series, and numerical tests define these successful cases. The results presented are useful for the intrinsic thermocouple problem and as background for more general heat conduction problems.**

## Nomenclature

$a$ = thermal diffusivity
$b$ = parameter defined by Eq. (10c)
$c$ = parameter defined by Eq. (15b)
$C$ = Schapery constant, = 2
$d$ = parameter defined by Eq. (17c)
$h$ = contact coefficient
$k$ = thermal conductivity
$q$ = heat flux
$\text{rerf}(z)$ = function defined by Eq. (11c)
$r_0$ = cylinder radius
$R_1$ = parameter defined by Eq. (17b)
$R_2$ = parameter defined by Eq. (17c)
$s$ = Laplace transform parameter
$t$ = time
$T$ = temperature
$T_0$ = initial temperature of body $A$
$\phi$ = influence function

*Superscripts*

$(\ )^*$ = Laplace transform
$(\ )^+$ = dimensionless quantity

*Subscripts*

$A$ = body $A$
$B$ = body $B$

## Introduction

THE unsteady surface element method (USEM) has been described previously for problems associated with thermally connected bodies.[1,2] When applicable, the USEM equations are more efficient than competing solution methods because the influence functions (i.e., kernels in Duhamel's integral) account for the presence of insulated or isothermal boundary conditions at noncontacting portions of the boundaries of the bodies.

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The USEM equations published to date use one node at the contacting surface, although transient multinode methods have been developed by Litkouhi[3]; Yovanovich[4] has considered the steady-state multinode case. The case of a single interface node is considered in this paper. This can represent a one-dimensional case or the average temperature for a two- or three-dimensional interface. Equations developed for a heat flux-based USEM procedure are used. Since the temperature-based procedure[1,2] can be utilized in a similar manner, this paper serves to illustrate both methods; for some problems the temperature-based method gives more accurate answers for late times.[1,2]

In this paper we first describe the USEM procedure yielding expressions in the Laplace transform domain. Next, a simplified version procedure suggested by Schapery[5] is employed to obtain the inverse Laplace transform of the USEM equations. Two different approximate relations are given, each of which produces functional forms of the solution. Finally, four test cases are considered that can be applied to design choices in the intrinsic thermocouple problem shown in Fig. 1 or similar problems.

## Analysis

Each of the geometries to be considered consists of two bodies, $A$ and $B$, with initial temperatures $T_0$ and 0, respectively. The intrinsic thermocouple problem shown in Fig. 1 contains the basic features. Imperfect contact is assumed at the interface.

The USEM equations are developed from a heat flux-based form of Duhamel's integral and a single node is on either side of the contacting surface; the equations for the $A$ and $B$ sides of the interface are

$$T_A(t) = T_0 - \frac{\partial}{\partial t}\int_0^t q(v)\,\phi_A(t-v)\,\mathrm{d}v \tag{1}$$

$$T_B(t) = \frac{\partial}{\partial t}\int_0^t q(v)\,\phi_B(t-v)\,\mathrm{d}v \tag{2}$$

where $q(t)$ is the heat flux leaving body $A$ and entering body $B$. The influence functions, $\phi_A(t)$ and $\phi_B(t)$, are the area-average temperature rises at the interface for a unit step heat flux into bodies $A$ and $B$, respectively.

For a contact coefficient $h$ between the two bodies, the heat flux $q(t)$ is related to the interface temperatures by

$$q(t)=h[T_A(t)-T_B(t)] \tag{3}$$

Expressions are found for $T_B(t)$ and $q(t)$ by taking the Laplace transform of Eqs. (1-3) to get

$$T_A^*(s)=T_0 s^{-1}-sq^*(s)\phi_A^*(s) \tag{4a}$$

$$T_B^*(s)=sq^*(s)\phi_B^*(s) \tag{4b}$$

$$q^*(s)=h[T_A^*(s)-T_B^*(s)] \tag{4c}$$

Solving Eqs. (4) simultaneously yields

$$q^*(s)=\frac{T_0}{s[s(\phi_A^*(s)+\phi_B^*(s))+h^{-1}]} \tag{5a}$$

$$T_B^*(s)=\frac{T_0\phi_B^*(s)}{s[\phi_A^*(s)+\phi_B^*(s)]+h^{-1}} \tag{5b}$$

Although these two equations can be numerically inverted in a number of different ways,[6] it is more desirable to have analytical inversion. For example, analytical expression, Eq. (A4) of Appendix A, could be used in place of the design charts developed by Wally[17] for the intrinsic thermocouple problem. The method is not universally applicable but works well for some problems where the usual inversion of the Laplace transform using tables can be very difficult. The method involves two rules: one for the direct Laplace transform (rule D) and one for the inverse transform (rule I).

The statement of rule D (direct) is

$$\psi^*(s)=\frac{1}{s}\psi(t)\Big|_{t=(Cs)^{-1}} \tag{6a}$$

and of rule I (inverse) is

$$\psi(t)=[s\psi^*(s)]\,|_{s=(Ct)^{-1}} \tag{6b}$$

These are approximate but Eq. (6b) is exact according to a Tauberian Theorem[7] for $s\to\infty$ (corresponding to small $t$) and for $s\to 0$ (corresponding to large $t$). The expression given by Eq. (6b) was suggested by Schapery[5,8] with the constant $C$ equal to 2. Equation (6b) is the first term of a series expression[16] for inverting the Laplace transform. The direct expression given by Eq. (6a) is obtained by rearranging Eq. (6b). Thus, Eqs. (6a) and (6b) are consistent with each other. Even though both expressions are not mathematically rigorous, utilization of them to obtain results of engineering accuracy is investigated in this paper.

One step in the inversion of Eqs. (5) is accomplished by substituting for $\phi_A^*$ and $\phi_B^*$ utilizing Eq. (6a) to get typically

$$s\phi_A^*(s)=\phi_A(t)\,|_{t=(Cs)^{-1}} \tag{7}$$

Then Eqs. (5) become

$$q^*(s)=\frac{T_0}{s[\phi_A(t)\,|_{t=(Cs)^{-1}}+\phi_B(t)\,|_{t=(Cs)^{-1}}+h^{-1}]} \tag{8a}$$

$$T_B^*(s)=\frac{T_0 s^{-1}\phi_B(t)\,|_{t=(Cs)^{-1}}}{\phi_A(t)\,|_{t=(Cs)^{-1}}+\phi_B(t)\,|_{t=(Cs)^{-1}}+h^{-1}} \tag{8b}$$

Next, rule I, Eq. (6b), is used. Equations (5) are first multiplied by $s$ which cancels with the $s$ in each equation. Then replacing $s$ in $(Cs)^{-1}$ by $(Ct)^{-1}$ means that $t$ is simply

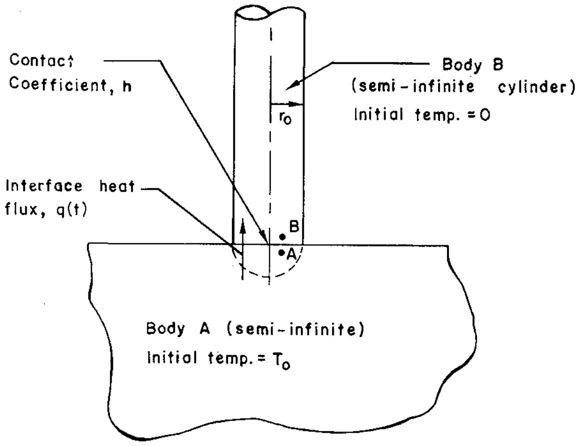


**Fig. 1 Intrinsic thermocouple problem.**

replaced by itself. The resulting equations are

$$q^D(t)=T_0[\phi_A(t)+\phi_B(t)+h^{-1}]^{-1} \tag{8c}$$

$$T_B^D(t)=T_0\phi_B(t)[\phi_A(t)+\phi_B(t)+h^{-1}]^{-1} \tag{8d}$$

In this solution both $D$ and $I$ were utilized and the superscript $D$ denotes "dual."

Another set of expressions can be given when the Laplace transforms $\phi_A^*$ and $\phi_B^*$ are available but the real-time functions $\phi_A(t)$ and $\phi_B(t)$ are difficult to obtain. In this method only rule I, Eq. (6b), is used to get

$$q^I(t)=T_0 Ct\left[\phi_A^*\left(\frac{1}{Ct}\right)+\phi_B^*\left(\frac{1}{Ct}\right)+\frac{Ct}{h}\right]^{-1} \tag{9a}$$

$$T_B^I(t)=\frac{T_0\phi_B^*(1/Ct)}{\phi_A^*(1/Ct)+\phi_B^*(1/Ct)+Ct/h} \tag{9b}$$

Unlike Eqs. (8c) and (8d), Eqs. (9a) and (9b) depend upon $C$.

## Applications

The results in this section provide analytical expressions that supplement the design figures plotted in Wally.[17] The quasicoupling method was used to develop the figures in Wally,[17] but the associated analytical solutions are not in the open literature. Quasicoupling is an early version of the USE method; the solution methods presented herein are more readily understood and are more powerful.

In this section four test problems are considered. Three of the problems have both bodies infinite in extent in the direction of the heat flow and one problem involves a lumped body and a semi-infinite body. The simplified procedure appears to work well for all the test problems.

Most of the influence functions used in the following examples are given in Table 1.

### Semi-infinite Cylinder Attached to Semi-infinite Body with Connecting Hemisphere

An approximate model for the intrinsic thermocouple problem (see Fig. 1) proposed by Henning and Parker[9] uses a connecting hemisphere of radius $r_0$ inside the semi-infinite body. The semi-infinite body is initially at $T_0$ and the cylinder is at zero. The hemisphere has the special properties of infinite conductivity and zero specific heat. Though this is not the best model for this problem, it can serve to illustrate several points. The contact coefficient $h$ is infinite since the contact is considered perfect. The heat flow leaving the hemisphere of surface area $2\pi r_0^2$ enters the cylinder of area $\pi r_0^2$. Hence there is a factor of one-half in the heat flux at the hemispherical

**Table 1 Some influence functions ($t^+ \equiv at/r_0^2$, $s^+ \equiv sr_0^2/a$)**

1) Surface of a semi-infinite body

$$\phi = \frac{2}{k}\left(\frac{at}{\pi}\right)^{1/2} \qquad \phi^* = \frac{1}{k}\left(\frac{a}{s^3}\right)^{1/2}$$

2) Average of circular heated area on semi-infinite body[13]
a) Small times ($t^+ < 0.6$)

$$\phi = \frac{2}{k}\left(\frac{at}{\pi}\right)^{1/2} - \frac{2r_0}{\pi k} t^+, \quad \phi^* = \frac{1}{k}\left(\frac{a}{s^3}\right)^{1/2}\left[1 - \frac{2}{\pi}(s^+)^{-1/2}\right]$$

b) Large times ($t^+ > 0.6$)

$$\phi = \frac{r_0}{k}\left[\frac{8}{3\pi} - \frac{1}{\sqrt{4\pi t^+}}\right], \quad \phi^* = \frac{1}{k}\left(\frac{a}{s^3}\right)^{1/2}(s^+)^{1/2}\left[\frac{8}{3\pi} - (s^+)^{1/2}\right]$$

3) Infinite region outside a spherical void

$$\phi = \frac{r_0}{k}\{1 - \mathrm{rerf}[\sqrt{t^+}]\}, \quad \phi^* = \frac{1}{k}\left(\frac{a}{s^3}\right)^{1/2}\left[1 + \frac{1}{\sqrt{s^+}}\right]^{-1}$$

4) Lumped body of volume $V$ and surface area $A$

$$\phi = Aat/Vk \qquad \phi^* = Aa/(Vks^2)$$

surface compared to the cylinder. Then the influence function for outside the hemispherical region, $\phi_A(t)$, is one-half of that given in case 3 of Table 1 and the $\phi_B(t)$ function is case 1 of Table 1.

The exact temperature solution can be found using the $\phi^*$ functions in Eq. (5b) to get

$$T_B^*(s) = \frac{T_0 \dfrac{1}{k_B}\left(\dfrac{a_B}{s^3}\right)^{1/2}}{s\left\{\dfrac{1}{k_B}\left(\dfrac{a_B}{s^3}\right)^{1/2} + \dfrac{1}{2k_A}\left(\dfrac{a_A}{s^3}\right)^{1/2}\left[1 + \left(\dfrac{a_A}{r_0^2 s}\right)^{1/2}\right]^{-1}\right\}} \tag{10a}$$

$$= \frac{bT_0[s^{1/2} + a_A^{1/2}/r_0]}{s(s^{1/2} + ba_A^{1/2}/r_0)} \tag{10b}$$

$$b = \left[1 + \frac{k_B}{2k_A}\left(\frac{a_A}{a_B}\right)^{1/2}\right]^{-1} \tag{10c}$$

Using inverse Laplace transform tables, Eq. (10b) is readily inverted to the solution given in Ref. 9,

$$T_B(t^+) = T_A(t^+) = T_0\{1 - (1-b)\,\mathrm{rerf}[(b^2 t^+)^{1/2}]\} \tag{11a}$$

$$t^+ \equiv a_A t/r_0^2 \tag{11b}$$

$$\mathrm{rerf}(z) \equiv e^{z^2}\mathrm{erfc}(z) \tag{11c}$$

This USEM derivation is shorter, simpler, and more direct than the procedure utilized in Ref. 9. For small and large times, Eq. (11a) is approximated by

$$T_B(t) = bT_0[1 + 2(1-b)(t^+/\pi)^{1/2}], \quad t^+ \ll 1 \tag{11d}$$

$$T_B(t) = T_0[1 - (1-b)b^{-1}(\pi t^+)^{-1/2}], \quad t^+ \gg 1 \tag{11e}$$

The dual equation, Eq. (8d), can be used with the $\phi(t)$ functions given in Table 1 (case 1 and one-half of case 3) to get

$$T_B^D(t^+) = \frac{2b(t^+)^{1/2}T_0}{2b(t^+)^{1/2} + \pi^{1/2}(1-b)[1 - \mathrm{rerf}((t^+)^{1/2})]} \tag{12a}$$

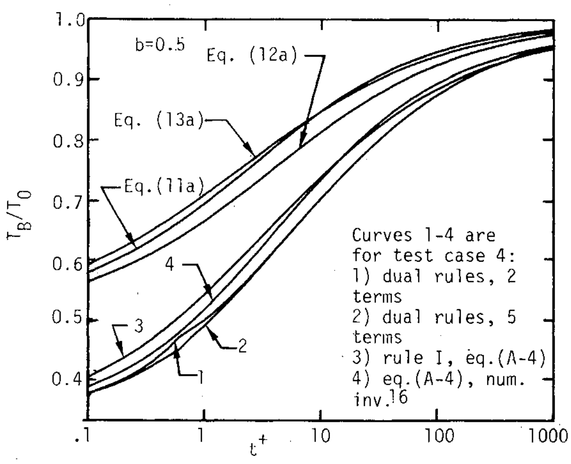


**Fig. 2 Curves for test cases 1 and 4.**

For small and large times, Eq. (12a) is approximated by

$$T_B^D(t^+) = bT_0[1 + 0.5(1-b)(\pi t^+)^{1/2}], \quad t^+ \ll 1 \tag{12b}$$

$$T_B^D(t^+) = T_0[1 - 0.5(1-b)b^{-1}(\pi/t^+)^{1/2}], \quad t^+ \gg 1 \tag{12c}$$

These expressions show that Eq. (12a) gives the same limiting values of $bT_0$ and $T_0$ for time 0 and $\infty$, respectively, given by Eq. (11a). In addition, the time, $t^+$, and $b$ dependences are the same for the first correction terms, but the coefficient $0.5\sqrt{\pi}$ in Eq. (12b) is 21% too low while the coefficient in Eq. (12c) is 57% too high. The actual values are much less in error, however, as can be noted from Fig. 2, which is for $b = 0.5$.

The rule I approximation is

$$T_B^I(t^+) = \frac{bT_0[1 + (Ct^+)^{1/2}]}{1 + b(Ct^+)^{1/2}} \tag{13a}$$

which can be approximated by

$$T_B^I(t^+) = bT_0[1 + 2^{1/2}(1-b)(t^+)^{1/2}], \quad t^+ \ll 1 \tag{13b}$$

$$T_B^I(t^+) = T_0[1 - (1-b)b^{-1}(2t^+)^{-1/2}], \quad t^+ \gg 1 \tag{13c}$$

These expressions show the correct behavior for the limiting cases but the coefficients are both 25% high. The error at maximum difference in Fig. 2 is less than 5%, which is frequently of acceptable accuracy.

Though the approximation to the exact Henning-Parker solution is in error, the error is much less than the effects of the imperfections in the model due to the imaginary infinite conductivity hemisphere. A better model for the intrinsic thermocouple problem is given as the last test case.

#### Semi-infinite Regions Connected by Spheres

A similar test case to the previous one is the heat flow between two semi-infinite bodies connected by an infinite-conductivity, zero-specific heat sphere. (A better model for the constriction resistance problem has been treated elsewhere.[2,4]) The perfect contact problem was solved by Heasley[10] using integral transforms. The heat flux is considered with imperfect contact, which was also investigated in Ref. 2. To simplify the presentation, the semi-infinite bodies are composed of the same material.

The influence functions are given by one-half of those of case 3 of Table 1, because the geometry can be considered as two hemispheres connected through an area of $\pi r_0^2$, which is the area of interest for the contact problem. Employing Eq.

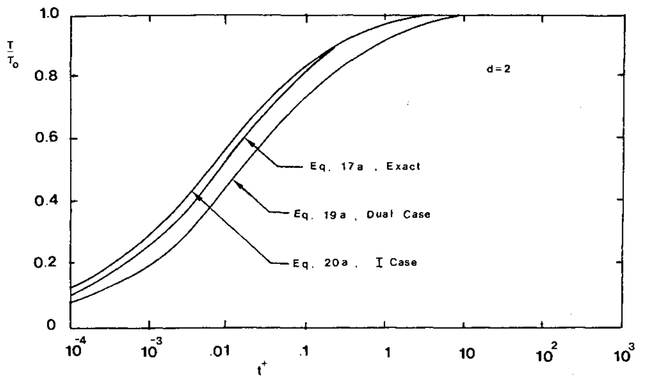


**Fig. 3 Curves for test case 3.**

(5a) and rearranging, one finds

$$q^*(s)=\frac{hT_0\,[s^{1/2}+a^{1/2}r_0^{-1}]}{s[s^{1/2}+a^{1/2}r_0^{-1}+a^{1/2}hk^{-1}]} \tag{14}$$

which has the exact solution of

$$\frac{q(t^+)}{hT_0}=c^{-1}\{1+(c-1)\,\mathrm{rerf}[(c^2t^+)^{1/2}]\} \tag{15a}$$

$$c\equiv 1+hr_0/k \tag{15b}$$

$$t^+\equiv at/r_0^2 \tag{15c}$$

An approximate solution is obtained by using the dual rule, Eq. (8c), to get

$$\frac{q^D(t^+)}{hT_0}=c^{-1}\{1-(1-c^{-1})\,\mathrm{rerf}[(t^+)^{1/2}]^{-1}\} \tag{16a}$$

For the I rule approximation given by Eq. (9a), one obtains

$$\frac{q^I(t^+)}{hT_0}=[1+(2t^+)^{1/2}]\,[1+c(2t^+)^{1/2}]^{-1} \tag{16b}$$

Both Eqs. (16a) and (16b) agree very well with the exact solution. The inaccuracy is less than 1% of full scale for $c=2$.

#### Lumped Sphere in an Infinite Medium

A case for which the simplified inversion is less accurate involves an infinite conductivity sphere at the initial temperature of zero in an infinite medium with an initial temperature of $T_0$. The sphere has finite heat capacity. This problem has been considered by Amos.[11] In Ref. 12 a related problem was applied to the determination of thermal properties. The quantity of interest is the temperature at the interface $r=r_0$ for the case of perfect contact.

The influence function, $\phi_A(t)$, is given in case 3 of Table 1 and $\phi_B(t)$ by case 4 with $V=4\pi r_0^3/3$ and $A=4\pi r_0^2$. A factor of two is not needed in this case because there is only one surface area of interest. The exact solution can be obtained by inverting Eq. (5b) to get

$$T(t^+)=T_0-\frac{T_0}{R_2-R_1}\{R_2\,\mathrm{rerf}[2R_2(t^+)^{1/2}]$$

$$-R_1\,\mathrm{rerf}[2R_1(t^+)^{1/2}]\} \tag{17a}$$

$$R_1=d+[d^2-d]^{1/2},\quad R_2=d-[d^2-d]^{1/2} \tag{17b}$$

$$d=\frac{3}{4}\left(\frac{k}{a}\right)_A\left(\frac{a}{k}\right)_B \tag{17c}$$

and $t^+$ is defined by Eq. (11b) and $d>1$. The small and large time behaviors of Eq. (17a) are given, respectively, by

$$T(t^+)=T_0 8\pi^{-1/2}d(t^+)^{1/2},\qquad t^+\ll 1 \tag{18a}$$

$$T(t^+)=T_0\{1-[8d(\pi(t^+)^3)^{1/2}]^{-1}\},\qquad t^+\gg 1 \tag{18b}$$

The dual rule, Eq. (8d), can be used to get

$$T^D(t^+)=\frac{T_0 4dt^+}{4dt^++1-\mathrm{rerf}[(t^+)^{1/2}]} \tag{19a}$$

$$T^D(t^+)=T_0 2d(\pi t^+)^{1/2},\qquad t^+\ll 1 \tag{19b}$$

$$T^D(t^+)=T_0[1-(4dt^+)^{-1}],\qquad t^+\gg 1 \tag{19c}$$

The functional form of Eq. (19b) is exactly the same as Eq. (18a), but the coefficient $2\sqrt{\pi}$ is 21% lower than that in Eq. (18a). The error is more serious in Eq. (19c) because the functional form in terms of $t^+$ is different, $(t^+)^{-3/2}$ in Eq. (18b) and $(t^+)^{-1}$ in Eq. (19c); the $d$ dependence is the same in both, however.

When rule I is used, Eq. (9b), one gets

$$T^I(t^+)=T_0[1+(Ct)^{1/2}]\,[1+(Ct^+)^{1/2}$$

$$+d^{-1}(16Ct^+)^{-1/2}]^{-1} \tag{20a}$$

$$T^I(t^+)=T_0 4d(Ct^+)^{1/2},\qquad t^+\ll 1 \tag{20b}$$

$$T^I(t^+)=T_0\left[1-\frac{1}{4dCt^+}\right],\qquad t^+\gg 1 \tag{20c}$$

Equation (20b) also has the correct functional form as does Eq. (19b), but the coefficient in Eq. (20b) is 25% too large. The expression given by Eq. (20c) is the same as the approximate result given by Eq. (19c) except for a factor of 2.

A comparison of the three approximate solutions is provided by Fig. 3. The maximum error is −8% of full scale for the dual approximation and about +3% for the I approximation for $d=2$. The errors are not large and the approximate curves predict the general behavior.

#### Intrinsic Thermocouple Problem

A better physical model for the intrinsic thermocouple problem than the Henning-Parker model is a cylinder attached to a semi-infinite body with a spatially constant heat flux between them.[1] Beck[13] found the influence function for the average temperature over a disk-shaped area exposed to a constant heat flux. An infinite series solution was given, only two terms of which are given in Table 1; expressions are given for early and late times.

Results are given for four approximations for the disk source influence function: 1) dual rule with the two terms of case 2 given in Table 1, 2) dual rule with enough terms for the early and late time expressions to match closely,[13] 3) rule I with the exact Laplace transform, and 4) accurate numerical inversion[16] of the exact Laplace transform. The influence function for the cylinder is given as case 1 of Table 1. The exact Laplace transform for the third and fourth approximations mentioned above is Eq. (A4) of Appendix A. A comparison of these four solutions is provided by Fig. 2 for $b=0.5$; all the results are within 5% of full scale and curves 1-3 are within 2% of curve 4, the most accurate. Notice that the Henning-Parker model solutions are much higher.

## Discussion

Two simple procedures are given for inverting the Laplace transform domain USEM equations. Some cases are given that demonstrate the accuracy can be good, better than 4%. For some models that are imperfect, the errors in the approximate method can be much less than in the model itself; for example, all curves in Fig. 2 are for the same general problem with the spread between curves indicating modeling and approximation errors. Based on the results to date, the early and late behavior is predicted quite well for cases when the correct functional form is predicted.

One case was found for which the procedures did not yield the correct functional form at late times, even so the values of calculated temperature were not greatly in error. If the results of the dual and I rules are averaged, the errors tend to be quite small, on the order of $\pm 3\%$.

In this paper only analytical solutions are used. There are many numerical methods for inverting Laplace transforms, one of which is the Gaver-Stehfest method[16] which has given excellent accuracy for USEM-type problems. This numerical procedure can be used to check the functional forms produced by the simplified method, and vice versa. This is more practical than a Tauberian test discussed in Appendix B. The numerical procedure complements the series expansion tests.

Both simplified and numerical inversion permit the use of more realistic models. Effects such as heat loss off the sides of the cylinder in the intrinsic thermocouple problem and a lumped body at the interface between two bodies slow the response of the system as does an interface resistance. These effects complicate the problem of finding the exact inverse transforms but fortuitously make the simplified equations more valid. More experience is needed with these techniques for other cases, particularly involving finite bodies. Another promising area is utilization of the simplified inversion techniques for the multinode case.[3]

## Appendix A

This Appendix gives the exact $\phi^*(s)$ for the average $T$ over a uniformly heated disk area on the surface of a semi-infinite body. The exact expression for the average dimensionless $T$ for an instantaneous disk source[13] is

$$f(t^*)=(\pi t^+)^{-1/2}\left\{1-e^{-1/2t^+}\left[I_0\left(\frac{1}{2t^+}\right)+I_1\left(\frac{1}{2t^+}\right)\right]\right\} \quad \text{(A1)}$$

The Laplace transform of Eq. (A1) can be expressed as

$$f^*(s)=s^{-1/2}-\frac{1}{\pi^{1/2}s^{3/4}}\int_0^\infty\frac{u^{3/4}J_{3/2}[(4us)^{1/2}]}{u^{3/2}(u+1)^{1/2}}\mathrm{d}u \quad \text{(A2)}$$

where Eq. (19), p. 5, and Eq. (12.3.9), p. 75, of Ref. 14 are utilized. If the substitutions $u=v^2$ and $s=w^2/4$ are made, this expression is recognized to be a Hankel transform that can be evaluated using Eq. (8.5.16) of Ref. 15 to yield

$$f^*(s)=s^{-1/2}-s^{-1}[I_1(2s^{1/2})-L_1(2s^{1/2})] \quad \text{(A3)}$$

where $L_1(\cdot)$ is the modified Struve function of order one.

The influence function is needed for a constant heat flux rather than an instantaneous source; that is, the integral of the instantaneous source is needed. Using well-known properties of Laplace transforms, this is obtained by simply dividing Eq. (A3) by $s$. In addition, compatibility is needed with the other influence functions that have units. Incorporating these changes in Eq. (A3) yields

$$\phi^*(s)=\frac{r_0}{k}\left\{\left(\frac{r_0^2s}{a}\right)^{-3/2}-\left(\frac{r_0^2s}{a}\right)^{-2}S\left[2\left(\frac{r_0^2s}{a}\right)^{1/2}\right]\right\} \quad \text{(A4)}$$

$$S(u)=I_1(u)-L_1(u) \quad \text{(A5)}$$

## Appendix B

This Appendix gives a Tauberian test that indicates when simplified inversion can be applied. The test works for the problems in this paper but it does not necessarily work for other problems. It is presented because the test is more quantitative and mathematically rigorous than the condition of a "slowly varying" derivative in Laplace transform space suggested by Sternberg.[8]

The accuracy of the approximate functional form compared to the exact solution is good for Eq. (17a), better for Eq. (11a), and best for Eq. (15a). That is somewhat surprising since all three exact solutions have similar growth with time specified by different linear combinations of the rerf function. It is postulated that simplified inversion works when the function satisfies the growth condition

$$f(t)\le t^{I-1/2} \quad \text{(B1)}$$

which is a slower growth rate than in the growth condition

$$f(t)\le e^{pt} \quad \text{(B2)}$$

associated with the Tauberian theorems. Conditions (B1) and (B2) are sufficient for the non-negative functions considered in this paper, although Tauberian condition (B2) can be extended to apply to more general functions.[7] By manipulating Eq. (B1) it can be shown that in Laplace transform space it becomes the condition

$$\frac{-s^{1/2}}{f^*(s)}\frac{\mathrm{d}}{\mathrm{d}s}[s^{1/2}f^*(s)]\le I \quad \text{(B3)}$$

For both Eqs. (11a) and (15a) the smallest integer assignable to $I$ is one for any value $s$. For Eq. (17a) the smallest value assignable to $I$ is two for midrange values of $s$. Apparently the accuracy of simplified inversion decreases as the value of the smallest integer assignable in condition (B3) increases.

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